20200114

Now on to permutations. We've all seen the word "permutation" before, usually in the context of counting the number of different linear arrangements of n distinct objects. When we do that calculation, we ignore the details of the individual permutations we are counting. For the next couple of classes we are going to define the concept of a permutation precisely, using our established understanding of relations and functions. We'll discuss the rudiments of a system of mathematics in which permutations are the fundamental objects. The idea of creating meaningful mathematical systems for things that are not numbers is of fundamental importance in discrete mathematics.

Definition: A **permutation** is a bijection from a set to itself.

For example, let the set A = {a, red, 3, α} One permutation of A is the bijection defined by the ordered pairs { (a,3), (red,a), (3, α), (α, red) } --- make sure that you agree that this is a bijection.

Note that the set on which the permutations are based **does not need** to be a set that has a natural order (such as {1,2,3,4} or {a,b,c,d}). In fact he set can be anything – the set A in the example just given is a demonstration of that: there is no natural order for this set, but we can still define permutations of it ... in fact there are 4! permutations of this set, because that is the number of bijections from A to itself.

However most of our representations of permutations are based on the idea of choosing some particular order of the elements of the set as the "normal" or "natural" or "agreed-on" order of the set – then we describe permutations based on how they differ from the natural order of the set.

When we are studying permutations the objects in the set don't usually matter – all that really matters is the size of the set. For this reason, when we talk about permutations the set A is usually just {1, 2, 3, ..., n} for some value of n. This is handy because we don't have to think too hard to come up with a natural order of the set!

We use S_n to represent the set of all permutations of the set {1, 2, 3, ..., n}

One of the first questions we can ask is, what is $|S_n|$? We already know the answer: The number of ways to create an ordered pair (1, x) (where x represents an element of {1, 2, ..., n}) is n. For each of those there are n-1 ways to create an ordered pair (2, y) ... and so on. The total number of bijections we can build is n!, so $|S_n| = n!$

Consider the permutation of {1,2,3,4} defined by { (1,4), (2,1), (3,3) (4,2) } Notice that under this function, 3 maps to itself. This is perfectly fine. In fact, there is a permutation that changes nothing: f(x) = x for all x. For {1,2,3,4} the ordered pairs for this permutation are {(1,1), (2,2), (3,3), (4,4)}. This is called the **identity permutation**, and we represent it with the Greek letter **iota** which looks like this: ι. It's basically *i* without the dot.

In fact we almost always use Greek letters to name permutations : π (pi), σ (sigma), and τ (tau) are among the favourites.

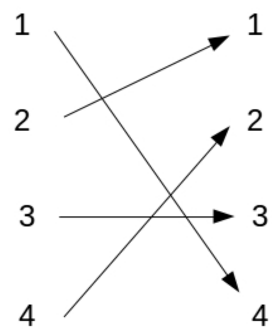
Permutations can be represented in a variety of ways. So far we have just listed the ordered pairs, but we can also use an n-by-n matrix, a diagram that shows the mapping of the set onto itself, or a 2-by-n matrix. For example, the permutation { (1,4), (2,1), (3,3) (4,2) } can also be represented as

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

in which each row corresponds to the first element in one of the ordered pairs, and each column corresponds to the second element. A "1" in the matrix indicates that the elements represented by the row and the column form an ordered pair. For example, there is a "1" in the **second** row and **first** column, so we know (2,1) is one of the ordered pairs in the permutation.

As was mentioned in class, we could also use the columns to represent the first elements of the pairs and the rows to represent the second elements of the pairs. This would transpose the matrix.

We can also draw a diagram to represent the permutation.



The 2-by-n matrix representation of this permutation looks like this:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

in which each column represents one of the ordered pairs in the permutation.

It's important to understand that each of these representations contains exactly the same information (they define the same permutation) and that if we are given any one of them we can construct all the others.

If we look at the 2-by-n matrix representation for different members of S_n such as

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

we can see that the first line is always the same. So we can leave it out! We represent those permutations by

$$\begin{bmatrix} 4 & 1 & 3 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 4 & 1 & 2 \end{bmatrix}$$

I will call this the **standard notation** for a permutation of {1, ..., n} because it is used very widely ... but as we will see, there is another notation that is often more useful in practice.

Remember that a permutation is a function, so we can use it as one ... the "input" is a position, and the "output" is the value that occupies that position. So if $\pi = \begin{bmatrix} 4 & 1 & 3 & 2 \end{bmatrix}$, we can say $\pi(1) = 4$, $\pi(4) = 2$, etc.

Composing permutations is just like composing other functions. If π and σ are permutations of {1, ..., n} we can write $\pi \circ \sigma$ to represent the result of applying σ (as a function) and then applying π

For example, let $\pi = \begin{bmatrix} 4 & 1 & 3 & 2 \end{bmatrix}$ and $\sigma = \begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix}$ ... what is $\pi \circ \sigma$?

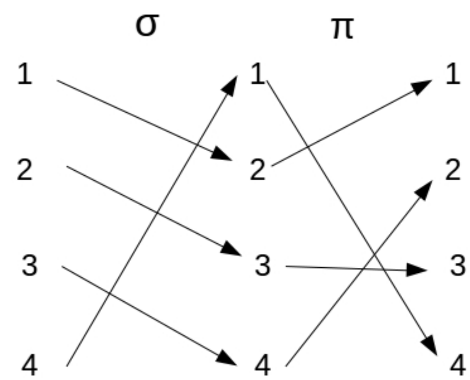
We can work it out: $\pi \circ \sigma(x) = \pi(\sigma(x))$, so we get

$$
\begin{aligned}
\pi \circ \sigma(1) = \pi(\sigma(1)) = \pi(2) &= 1 \\
\pi \circ \sigma(2) = \pi(\sigma(2)) = \pi(3) &= 3 \\
\pi \circ \sigma(3) = \pi(\sigma(3)) = \pi(4) &= 2 \\
\pi \circ \sigma(4) = \pi(\sigma(4)) = \pi(1) &= 4
\end{aligned}
$$

and look ... the result is a permutation! Exercise: Try to prove that the composition of two permutations will always be a permutation.[1]

We can create a diagram to visualize the composition of permutations. Using the same two permutations π and σ as in the previous example we get the figure on the next page:

1 Hint: prove a broader statement: the composition of two bijections will always be a bijection. The result for permutations follows automatically since every permutation is a bijection.



This diagram illustrates $\pi \circ \sigma$. To see this, try starting at some position x in the first column (for example, 3) and follow the arrows to the last column (starting with 3, we end up on 2) ... and find that this corresponds exactly to $\pi \circ \sigma(x)$.

We can also just think of the operation of a permutation as "turns x into y", so we can interpret $\pi \circ \sigma(3) = 2$ as "σ turns 3 into 4, then π turns 4 into 2"