Notation for Partial Ordering – a Mixed Blessing

We often use "$\leq$" to represent a generic relation that defines a partial ordering on a set – this is unfortunate because it can lead to confusion when "$\leq$" is also being used in its normal sense to compare numbers. On the other hand "*less than or equal*" is a great example of a relation that is reflexive, transitive and anti-symmetric so this symbol serves as a useful reminder of the essential properties of a poset.

With $\leq$ defined for a poset, we can define $<$, $>$, and $\geq$ for the poset as well:

Let x and y be elements of a poset where the relation is represented by $\leq$.

$x \geq y$	if	$y \leq x$
$x < y$	if	$x \leq y$ and $x \neq y$
$x > y$	if	$y < x$

and we can negate these ... for example

$x \not\leq y$ means x is not $\leq$ y

This is where some confusion can occur. When $\leq$ is being used in its "natural" sense to compare numbers, we know that if $x \not\leq y$ then $x > y$

But in an arbitrary poset, is it true that if $x \not\leq y$, then we can be sure that $x > y$? Not necessarily, because x and y may be **incomparable**. (See the definition of incomparable below, but for now it just means exactly what it sounds like: x and y are not related in this relation.)

Maxima, Minima, Maximum, Minimum

Let P = (S, $\leq$) be a poset based on set S with relation $\leq$. (Remember, we **cannot** assume that S is a set of numbers, or that $\leq$ has its normal arithmetical meaning. All that we know is that $\leq$ represents a relation that is reflexive, transitive, and anti-symmetric.)

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Note: a poset may have many maximal and many minimal elements, but cannot have more than one maximum nor more than one minimum element.

Note: a maximum element must also be maximal, but a maximal element may not be a maximum (and the parallel relationship holds for minimum and minimal elements)

Note: a poset may not contain either a maximum or a minimum element. For example, consider the relation on the set $S = \{a, b\}$ consisting of the ordered pairs $\{(a, a), (b, b)\}$. First, make sure that you agree that this is a poset. Second, convince yourself that a and b are both maximal and both minimal, but neither of them is either maximum or minimum.

Comparability, Chains, Anti-Chains, Width and Height

Definition: In a poset P, x and y are **comparable** if either $x \leq y$ or $y \leq x$. x and y are **incomparable** if neither $x \leq y$ nor $y \leq x$

Definition: In a poset P, a **chain** is a set of values that are **all** mutually comparable.

That is, $C \subseteq S$ is a chain if $\forall\ x \text{ and } y \in C$, x and y are comparable.

Definition: In a poset P, an anti-chain is a set of values in which all the distinct values are mutually incomparable.

That is, $A \subseteq S$ is an anti-chain if $\forall\ x \text{ and } y \in A \text{ with } x \neq y$, x and y are incomparable.

Definition: In a poset P, the **height** is the size of the largest chain

Defintion: In a poset P, the **width** is the size of the largest anti-chain

For example, consider the "divides" poset that we looked at earlier. It has height 4 and width 6. As an exercise, find at least one chain and at least one anti-chain that match these answers.

Theorem: Let P = (S, $\leq$) be a finite poset (i.e. S is a finite set). Then P has at least one maximal element, and P has at least one minimal element.

Proof: Let C be the largest chain in P. (If two or more are tied, pick either one.) We know C is finite since C $\subseteq$ S. All the elements of C are comparable. We will now identify an element x in C such that no element y in C satisfies x < y

Let m be any element in C. If there is no element y in C such that m < y, let x = m. Otherwise, let m′ be an element in C such that m < m′. Now if there is no y in C such that m′ < y, let x = m′. Otherwise, let m′′ be an element in C such that m′ < m′′, and continue in this way. Since C is finite, we eventually find an element x such that no element y in C satisfies x < y.

Now suppose there exists z in S such that x < z. Then C′ = C $\cup$ {z} is a set of comparable elements, and |C′| > |C| ... which contradicts our choice of C as the largest chain.

Therefore there is no z in S such that x < z. Therefore m is maximal.

We can use a completely parallel argument to show that there is a minimal element.

Linear Orders

The defining feature of partial orders is that some of the elements may be incomparable. However, it is certainly possible that in some reflexive, transitive and anti-symmetric relations all elements **could be** comparable to each other.

For example consider the set S = {1,2,3,4,5} and let "$\leq$" be normal "less-than-or-equal", and let A = (S, $\leq$). It is easy to show that A is a partial order because it satisfies the **reflexive, transitive and anti-symmetric** properties.

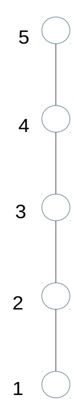
Let x and y be any elements of S. Clearly $x \leq y$ or $y \leq x$ since they are just numbers and this is a property of numbers. Therefore all pairs of values in S are comparable.

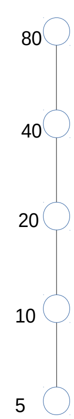
A partial order in which all pairs of elements are comparable is called a **total order** or a **linear order.**

Let's look at another linear order. Let T = {5, 10, 20, 40, 80} and let the relation be "divides", for which we will use the normal symbol "|" (remember, a | b means a **divides into** b, so 5 | 10 is true, but 10 | 5 is false)

I claim that B = (T, |) is a linear order. You should work through the details.

Consider the Hasse diagrams for these two linear orders:





Looking at these, you might notice a certain similarity ... in fact, they are **identical** except for the labels on the elements! We have a word for that: **isomorphism.** (The word "isomorphism" comes from "iso" meaning "same" and "morph" meaning "shape")

Loosely, we can say that any two posets are isomorphic if their Hasse diagrams look identical when you take the labels off ... but that's pretty loose. We would also have to talk about mirror-images etc. to account for flipping all or part of a Hasse diagram from one side to the other. It's better if we have a mathematical definition of isomorphism for posets, and we do.

Definition: Let $P = (S_1, \leq)$ and $Q = (S_2, \preceq)$ be any two posets (I have used different symbols for the relations to show that we don't have to use the same relation in both posets).

Then P and Q are **isomorphic** (which we write $P \cong Q$) if
$\exists$ a bijection f: $S_1 \rightarrow S_2$ such that $x \leq y$ if and only if $f(x) \preceq f(y)$ $\forall$ x, y $\in S_1$

We call such a bijection **order-preserving** (because that is what it does).

Basically this means that in order to show that two posets are isomorphic (which is useful to know because it means that everything that is true about one of them is also true about the other) we just have to show that there is a mapping between the elements that preserves order – so if two elements are related in the first poset, the elements they are mapped onto in the second poset must be related in the same way.

Let's use the posets A and B that we defined earlier to try out this new isomorphism thing. To show that $A \cong B$ we just have to find an order-preserving bijection. The similarity of the Hasse diagrams pretty much tells us what the bijection should be:

x	f(x)
1	5
2	10
3	20
4	40
5	80

But we still have to prove that f() is an order-preserving relation. In this example that is made easy by the fact that we can see a pretty simple formula for computing f(x): $f(x) = 5 * 2^{x-1}$ (If you are wondering where that came from, I had it in mind when I set up these posets. Normally coming up with the proper bijection takes some effort.) Let x and y be elements of $\{1, 2, 3, 4, 5\}$. We need to show $x \leq y$ if and only if $f(x) \mid f(y)$.

Suppose $x \leq y$. That means $y = x + k$ for some $k \geq 0$

Now $f(x) = 5 * 2^{x-1}$ and $f(y) = 5 * 2^{y-1}$

$= 5 * 2^{x+k-1}$

$= 5 * 2^{x-1+k}$

$= 5 * 2^{x-1} * 2^k$

$= f(x) * 2^k$

so $f(x) \mid f(y)$

Now suppose $f(x) \mid f(y)$ This means $5 * 2^{x-1} \mid 5 * 2^{y-1}$

$\Rightarrow \quad 2^{x-1} \mid 2^{y-1}$

$\Rightarrow \quad x - 1 \leq y - 1$

$\Rightarrow \quad x \leq y$

So f is order-preserving and we are done.