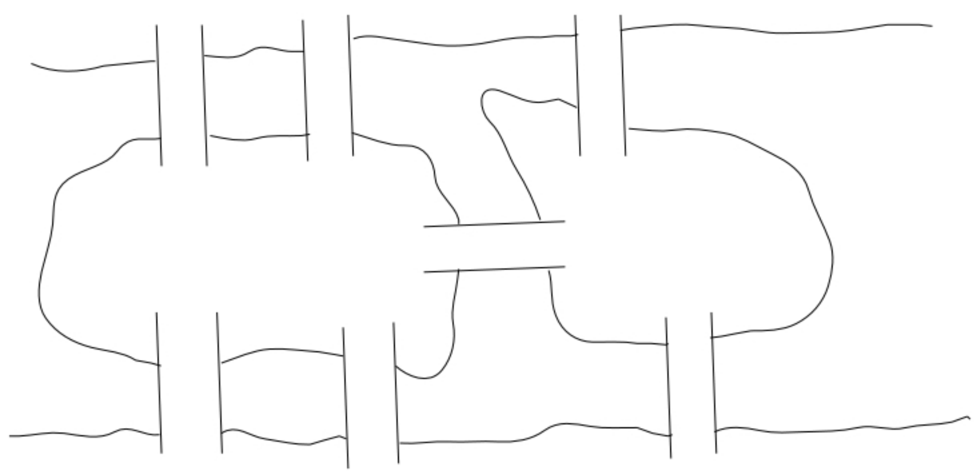
Graph Theory

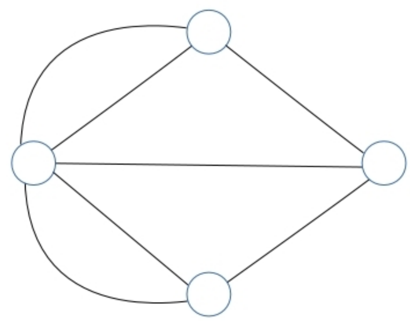
Here is a (not very) detailed and (not very) topographically accurate map of the city of Konigsberg (now called Kaliningrad) as it was in the year 1736. As you can see, the river Pregel runs through the middle of the city, and two islands sit in the middle of the river. The islands are connected to each other and to the banks of the river by seven bridges.



The citizens of Konigsberg were consumed by an apparently unsolvable problem: was it possible to leave one's home, cross each bridge exactly once and return home again? It is said that many people claimed to have done it but nobody could quite remember how.

In 1736 Leonhard Euler put an end to the raging controversy by proving that it was not possible – in fact it was not even possible to go for a walk that started *anywhere* and ended *anywhere* and crossed each bridge exactly once. In the process of proving this, Euler invented graph theory.

Here's how he did it. He simplified the map to look like this:



The circles (which could be squares, triangles, dots, whatever) are called **vertices** (the singular form is **vertex**) and the lines are called **edges**. Vertices are sometimes called **points**.

Two vertices that are joined by an edge are said to be **adjacent**. They are also called **neighbours**.

Here is a proof of Euler's result.

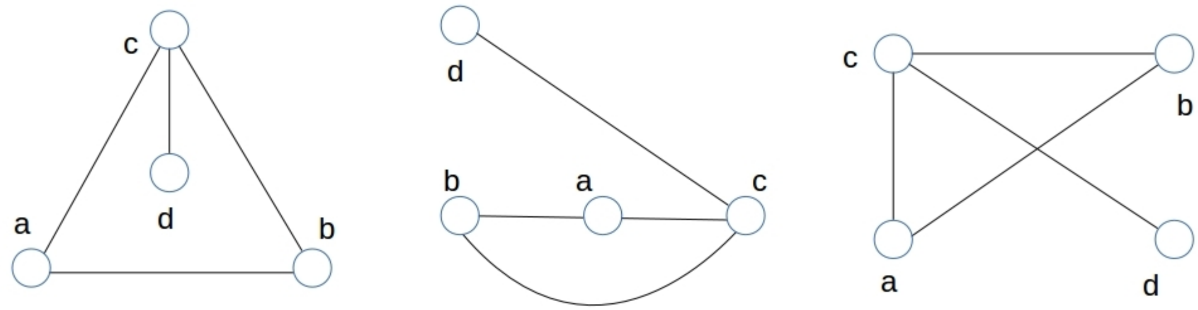
Suppose it were possible to start at any vertex, follow an edge to another vertex, then follow another edge to another vertex, and so on until all edges had been used exactly once, and end up back where we started. Consider the ***second*** vertex we visit: every time we "arrive" at this vertex we have to leave again. This means that this vertex must have an even number of edges touching it. But in the diagram (and also on the map) we can see that every vertex has an odd number of edges touching it. This means no vertex can be the second one in the sequence ... which is not possible.

Now for some formal definitions:

A **graph** consists of a set of vertices V and a set of edges E, where E is a set containing two-element subsets of V. Sometimes we describe edges as *unordered pairs* of vertices.

We often think of a graph as a drawing, but it is important to remember that the graph is defined by the sets V and E, not by a particular drawing of them. A graph can be drawn in many different orientations and sizes, but it is still the same graph.

For example the graph G=(V,E) where V = {a,b,c,d} and E = { {a,b}, {a,c}, {b,c}, {c,d} }
can be drawn many ways, including these:



This definition of "graph" is the one used in our textbook. It places a couple of implicit restrictions on the set of edges:

- all edges must be distinct – because E is a set, no edge can be duplicated
- no edge can join a vertex to itself – because each element of E is a *two-element* subset of V
- each edge is unordered – so an edge joining vertex x to vertex y can be written either as {x,y} or as {y,x} .... note that sometimes edges are written using simple parentheses, such as (x,y) or (a,b)

Note that by this definition, Euler's diagram – the first graph ever – is not a graph because it has multiple edges with the same end-vertices.

Most mathematicians when writing about graph theory define graphs in a way that allows multiple edges between vertices (often called *parallel edges*) and edges that join vertices to themselves (often called *loops*) ... but then go on to say "but we will only consider graphs without parallel edges and loops". In other words they define the more general class, but then restrict themselves to graphs as Scheinerman defines them. Our text just jumps directly to the definition that is most commonly used in practice.

For the record, most mathematicians would say that Scheinerman is defining **simple graphs**.

There are several more definitions and bits of notation to get out of the way.

The number of vertices in a graph G is sometimes called the **order** of G, and sometimes written as $v(G)$, and sometimes written as just the letter **n**. |V| is also used.

The number of edges in a graph G is sometimes called the **size** of G, and sometimes written as $e(G)$, and sometimes written as just the letter **m**. |E| is also used.

A **walk** in a graph is a sequence of edges such that the first edge shares a vertex with the second edge, the second edge shares its other vertex with the third edge, and so on. In a walk, vertices and edges may be used multiple times. Our text gives a different but equivalent definition: a **walk** is a sequence of vertices in which each vertex is adjacent to the previous vertex in the sequence.

A **path** is a walk that uses each edge and vertex at most once.

A **circuit** is a walk that returns to its starting vertex.

A **cycle** is a path that returns to its starting vertex (thus bending the path rule a little bit). Another way to define a cycle is "a path with end-vertices x and y, plus the edge {x,y}"

The **degree** of a vertex is the number of neighbours it has. We usually use **d(v)** to represent the degree of vertex v.

Two vertices are **connected** if there is a path that starts at one and ends at the other.

Theorem: Let G be a graph. Then $\sum_{v \in V} d(v) = 2 \cdot e(G)$

Proof: The left side is the sum of the degrees of the vertices of G. Every edge gets counted twice in this sum (for example, an edge {x,y} contributes 1 to the degree of x and 1 to the degree of y). Thus the sum of the degrees is twice the number of edges.

This has an interesting corollary:

Corollary: Let G be a graph. Then the number of vertices of G with odd degree is even.

Proof: Suppose G has an odd number of odd-degree vertices. Then $\sum_{v \in V} d(v)$ must be odd, which contradicts the theorem we just proved. Therefore G cannot have an odd number of odd-degree vertices.