Euler Tours and Hamilton Cycles

Let's revisit the problem that Euler solved for the city of Konigsberg, but now we can apply it to all graphs:

Given a graph G, does G contain a circuit that contains every edge exactly once?

Such a circuit is called an **Euler Tour**. A graph that has an Euler Tour is said to be **Eulerian**.

This is a very practical problem. For example, the graph might represent the streets of a city, and we might need to plan a route for the Recycling Service – they need to drive along every street, but it's a waste of time to drive along any street more than once. Ideally they want a route that lets them drive along every street exactly once. Google Street View vehicles and postal delivery services have the same goal. A company called HiBot is developing robots that will crawl through water-supply pipes to determine their integrity (https://www.constructiondive.com/news/this-amphibious-robot-can-crawl-through-pipes-to-collect-data/438168/) - planning a route that explores every pipe in a network with minimal repetition will be an important task.

Before discussing this problem, let's look at another, similar-looking problem:

Given a graph G, does G contain a cycle that contains every vertex? (Remember, the definition of a cycle precludes the possibility of including any vertex more than once.)

Such a cycle is called a **Hamilton Cycle,** and a graph with a Hamilton Cycle is said to be **Hamiltonian.** Hamilton was a famous mathematician and one of the pioneers of graph theory. He once marketed a board game based on finding Hamilton Cycles in graphs – for some reason it was not a million-seller.

This is also a very practical problem. If a delivery service needs to visit the same set of locations every day, the ideal route would be one that never needs to "double back" and visit some location twice on the way to other locations. I remember that several years ago some of the Computer Science professors at Simon Fraser University were engaged in finding a practical minimal-cost solution to this problem for a restaurant-service company in Vancouver.

The reason for introducing both of these questions at the same time is to think about the relative difficulty of answering them. It may seem that the Hamilton Cycle problem is easier because it only involves finding a permutation of the vertices, whereas the Euler Tour problem involves ordering all the vertices and all the edges – and since it is a circuit, each vertex may occur multiple times in the solution.

In fact, exactly the opposite is true: for an arbitrary graph G, the Euler Tour question can be answered extremely easily, whereas the Hamilton Cycle question is so difficult that most mathematicians believe that there will never be an efficient method for answering it.

This is one of the things that I find so fascinating about graph theory: it is full of counter-intuitive results.

You will have to wait until CISC-365 for evidence that the Hamilton Cycle problem is so difficult, but the easiness of the Euler Tour question is well within our scope.

Theorem: Let G be a connected graph. G is Eulerian if and only if every vertex has even degree.

Proof:

First, observe that if G is Eulerian, then every vertex must have even degree, since the tour must pair up the edges at each vertex into "arrive/depart" pairs.

Now we show that if G is connected and all degrees are even, then G is Eulerian.

We will use a minimal counter-example proof, based on the number of edges in the graph. For a base case, we will consider graphs with 3 edges. A bit of experimentation shows that there are very few connected graphs with 3 edges, and only one of them has all degrees even: K_3. (We use the notation K_n to name the graph on n vertices that has all possible edges.) This is also the only one that is Eulerian. So the claim is true for m = 3.

Suppose G is a counter-example, and let G have the smallest number of edges in any counter-example. As always, we will use m to represent the number of edges in G.

We can show G contains a cycle: we can just start following edges – eventually we will come back to some vertex we have visited before, because every time we "arrive" at a vertex we can leave on a different edge (because the degree is even) and the graph is finite ... so we can't keep visiting new vertices forever.

Let C be a cycle in G. Let G' be the graph that results in removing the edges of C. This reduces the degree of each vertex in C by exactly 2 – so G' is a graph in which each vertex has even degree, and G' has < m edges ... so G' is **not** a counterexample.

Suppose G′ is connected. Then G′ has an Euler Tour. We can extend that to an Eulerian tour of G by following the Tour of G′ until we reach one of the vertices in C ... then follow the edges of C back to that vertex ... then follow the rest of the Tour of G′. This completed tour includes every edge of G exactly once, so G **is** Eulerian (and therefore not a counter-example).

Now suppose G′ is not connected. Each of its connected components is too small to be a counterexample, so each one has an Euler Tour. I leave it as an exercise for you to determine how we can use these Euler Tours of the components of G′, plus the cycle C, to construct an Euler Tour of G. (Again, this means G is not a counter-example.)

Thus we find that the minimal counter-example is not a counter-example ... and this contradiction allows us to conclude that no counter-examples exist.