CISC-203*
Test #1
September 25, 2018

Student Number (Required) ______________________

Name (Optional) ________________________________

This is a closed book test. You may not refer to any resources.

This is a 50 minute test.

Please write your answers in ink. Pencil answers will be marked, but will not be reconsidered after the test papers have been returned.

The test will be marked out of 50.

Question 1	/10
Question 2	/10
Question 3	/10
Question 4	/10
Question 5	/10
TOTAL	/50

By writing my initials in this box, I give Dr. Dawes permission to destroy this test paper if I have not picked it up by January 15, 2019.

Question 1 : (10 marks)

Let $f : A \to B$ and $g : B \to C$ be functions, where A, B and C are finite sets.

Prove the following statement:

If f and g are both ***onto,*** then $|A| \geq |C|$

Use any valid proof technique:

Solution:

If $|A| < |B|$, then the Pigeonhole Principle tells us there are no onto functions from A to B. Therefore $|A| \geq |B|$

Similarly, $|B| \geq |C|$

Therefore $|A| \geq |C|$

Marking:

For a solution similar to the above It is not necessary to mention the PHP by name.	***10/10***
For a sound alternative solution (Proof by Contradiction, for example)	***10/10***
For a solution that takes a good approach (for example, induction is not ***a good approach here) but contains a significant error.***	***8/10***
For a solution that shows understanding of the question but goes off the rails due to poor	***6/10***

choice of proof technique

For a solution that shows understanding of the question (meaning of "function", meaning of "onto", etc.) but goes no further	*5/10*
For a solution that shows limited understanding of the question (for example, understanding "function" but not "onto")	*3/10*
For a solution that shows no understanding of the question.	*0/10*

Question 2 : (10 marks)

Prove the following using any valid proof technique:

$$\sum_{i=1}^{n}(2i-1) = n^2 \quad \forall \text{ integers } n \geq 1$$

Solution:

Direct proof:

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 1$$

$$= 2\sum_{i=1}^{n} i - n$$

$$= 2\frac{(n+1)n}{2} - n$$

$$= n^2 + n - n$$

$$= n^2$$

PBI:

Base case: n = 1 $\quad \sum_{i=1}^{1}(2i-1) = 2-1 = 1 = 1^2$

Inductive step: Suppose $\sum_{i=1}^{k}(2i-1) = k^2$ for some $k \geq 1$

Now consider k+1

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + 2(k+1) - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2\end{aligned}$$

PMCE :

Let k be the minimal counterexample.

Observe that $\sum_{i=1}^{1}(2i-1) = 1^2$, so $k \geq 2$

$\Rightarrow k-1 \geq 1$, so $k-1$ is not a c.e.

$$\Rightarrow \sum_{i=1}^{k-1}(2i-1) = (k-1)^2$$

$$\begin{aligned}\Rightarrow \sum_{i=1}^{k}(2i-1) &= (k-1)^2 + 2k - 1 \\ &= k^2 - 2k + 1 + 2k - 1 \\ &= k^2\end{aligned}$$

$\therefore k$ is not a c.e. CONTRADICTION

Marking:

For any complete proof (including proofs that contain trivial errors such as an incorrect subscript)	***10/10***
For a proof that shows good understanding of the chosen proof method but has a significant flaw	***8/10***
For a solution that shows understanding of the question but goes off the rails due to poor choice of proof technique	***6/10***
For a solution that shows understanding of the question (meaning of "summation", etc.) but goes no further	***5/10***
For a solution that shows limited understanding of the question	***3/10***
For a solution that shows no understanding of the question.	***0/10***

Question 3 : (10 marks)

(a) [3 marks] Find a value of k that makes the following statement true:

All integers $n \geq k$ can be written as $n = 3a + 5b$ where a and b are positive integers.

Solution:

The statement is true for $k = 16$

Marking:

For 16	***3/3***
For any integer > 16	***2/3***
For any integer < 16	***1/3***
For no answer	***0/3***

(b) [7 marks] Use either **Proof by Minimum Counter-Example** or **Proof By Induction** to prove that your value of k makes the statement true.

Solution:

PMCE:

Let k be the minimum counterexample

Observe that 16 = 3*2 + 5*2

17 = 3*4 + 5*1

18 = 3*1 + 5*3

$\therefore k \geq 19$

$\Rightarrow k - 3 \geq 16$

$\Rightarrow k - 3$ **is not a c.e.**

$\Rightarrow k - 3 = 3a + 5b$ **for some positive integers a and b**

$\Rightarrow k = 3(a + 1) + 5b$

$\Rightarrow k$ **is not a c.e. CONTRADICTION**

PBI:

Base cases:

16 = 3*2 + 5*2

17 = 3*4 + 5*1

18 = 3*1 + 5*3

Inductive step: Assume the statement is true for all values of n **in the range** $16 \leq n \leq k$ **for some** $k \geq 18$

Consider $n = k + 1$

$n \geq 19$ **(since** $k \geq 18$**)**

$\Rightarrow k \geq n - 3 \geq 16$

$\Rightarrow n - 3 = 3a + 5b$ **for some positive** a **and** b

$\Rightarrow n = 3(a + 1) + 5b$

$\therefore$ **the statement is true for** $n = k + 1$

$\therefore$ **the statement is true** $\forall \ n \geq 16$

Marking:

For any complete proof using either technique (including proofs that contain trivial errors such as an incorrect subscript)	***10/10***
For a proof that shows good understanding of the chosen proof method but has a significant flaw in the proof.	***8/10***
For a solution that shows understanding of the question but limited understanding of the chosen proof technique.	***6/10***
For a solution that shows understanding of the question (meaning of "summation", etc.) but goes no further	***5/10***
For a solution that shows limited understanding of the question	***3/10***
For a solution that shows no understanding of the question.	***0/10***

Question 4 : (10 Marks)

Let $\pi, \sigma,$ and θ be permutations in S_n

(a) [5 marks]

Prove that π^{-1} is unique. That is, prove that

if $\pi \circ \sigma = \sigma \circ \pi = \iota$ and $\pi \circ \theta = \theta \circ \pi = \iota$, then $\sigma = \theta$

Solution:

Assume $\pi \circ \sigma = \sigma \circ \pi = \iota$ and $\pi \circ \theta = \theta \circ \pi = \iota$

$\Rightarrow \pi \circ \sigma = \pi \circ \theta$
$\Rightarrow \pi^{-1} \circ \pi \circ \sigma = \pi^{-1} \circ \pi \circ \theta$
$\Rightarrow \sigma = \theta$

Marking:

For any complete solution (including, for example a solution using proof by contradiction, etc.)	***5/5***
For a proof that shows good understanding of the question (permutations, composition of permutations, inverses) but has a significant flaw in the argument	***3/5***
For a solution that shows limited understanding of the question	***1/5***
For a solution that shows no understanding of the question	***0/5***

(b) [5 marks]

We use π^x to represent $\underbrace{\pi \circ \pi \circ \cdots \circ \pi}_{x}$

For example, $\pi^3 = \pi \circ \pi \circ \pi$

Prove $\quad \forall x \geq 1, \qquad \pi^x = \pi^{x+1} \;\Rightarrow \pi = \iota$

Solution:

Suppose $\pi^x = \pi^{x+1}$ for some $x \geq 1$

By composing each side with π^{-1} x times, we reduce the left side to ι and the right side to π, giving the required result $\iota = \pi$

In notation

$$\underbrace{\pi^{-1} \circ \pi^{-1} \cdots \circ \pi^{-1}}_{x} \circ \underbrace{\pi \circ \pi \cdots \circ \pi}_{x} = \underbrace{\pi^{-1} \circ \pi^{-1} \cdots \circ \pi^{-1}}_{x} \circ \underbrace{\pi \circ \pi \cdots \circ \pi}_{x+1}$$

$$\Rightarrow \iota = \pi$$

Marking:

as for part (a)

Question 5 : (10 Marks)

We call a permutation π a ***derangement*** if $\pi(i) \neq i \ \ \forall\, i \in \{1, 2, \ldots, n\}$

For example

$\pi = [4\ 5\ 6\ 7\ 1\ 2\ 3]$ is a derangement

$\tau = [4\ 2\ 6\ 7\ 1\ 3\ 5]$ is not a derangement because $\tau(2) = 2$

(a) [5 marks]

Let π be a permutation of $\{1, 2, \ldots, n\}$ where $n \geq 2$

Prove : if π is **not** a derangement, the cycle representation of π **must** contain at least two cycles. Use any valid proof technique.

Solution:

Assume π is not a derangement. Then $\pi(i) = i$ for some i

In cycle representation for π, i forms a cycle of length 1: (i)

Therefore there must be at least one more cycle to contain the other elements of the permutation.

Marking:

as for Question 4

(b) [5 marks]

Prove or disprove this statement:

If π is a derangement, then π^{-1} is also a derangement

Solution:

Let π be a derangement

Recall that $\pi(x) = y$ means $\pi^{-1}(y) = x$ and vice versa (π^{-1} just undoes π)

Suppose π^{-1} is not a derangement.

$\Rightarrow \pi^{-1}(i) = i$ **for some i**

$\Rightarrow \pi(i) = i$

$\Rightarrow \pi$ **is not a derangement CONTRADICTION**

$\therefore \pi^{-1}$ **is a derangement**

OR

let π be a derangement

$\Rightarrow$ **the cycle representation of π has no cycles of length 1**

We know the cycle representation of π^{-1} just consists of the same cycles as π, reversed

$\Rightarrow$ **the cycle representation of π^{-1} has no cycles of length 1**

$\Rightarrow$ **there is no i such that $\pi^{-1}(i) = i$**

$\therefore \pi^{-1}$ **is a derangement**

Marking:

as for Question 4