Question 1: (10 Marks)

Define $\preceq$ on CISC courses by

$x \preceq y$ **iff** $x = y$ **or** x **is a prerequisite of** y

For example $203 \preceq 203$, $121 \preceq 124$, **etc**

Let S be the following (somewhat reduced) set of CISC courses, and let R be the following set of $\preceq$ connections.

S = { 102, 121, 124, 203, 204, 235 }

R = { (102, 203), (102, 204), (121,124), (121,203), (121, 204), (124, 235), (203, 235) }

(a) [6 marks] List all the pairs that need to be added to R to make (S, R) a poset

Solution: The missing pairs are
(102,102),(121,121),(124,124),(203,203),(204,204),(235,235),

(102,235),(121,235)

Marking: **3 marks for adding the reflexive pairs**
3 marks for adding the transitive pairs
part marks for knowing what types of pairs are missing, but not getting them right (2 if the error is small, 1 if the error is large)
0 for no answer or not understanding the question

(b) [4 marks] Determine the height and width of the poset

Solution: *The height is the size of the largest chain: 3*
The width is the size of the largest antichain: 3

Marking: **correct height: 2 marks**
knowing what height means, but getting the wrong answer: 1
correct width: 2 marks
knowing what width means, but getting the wrong answer: 1

Students are not required to state the meaning of height and width in order to get full marks

Question 2 : (15 marks)

Let $S = \{1, 2, 3, 4\}$ and let F be the set of all functions from S to S.

Note that F contains functions which are bijections as well as functions that are not.

We define the relation $\preceq$ on F as follows:

Let f and g be functions in F. $f \preceq g$ **iff** $f(i) \leq g(i) \quad \forall\, i \in S$

(where $\leq$ has the standard "less-than-or-equal" meaning)

For example, if f and g are given by this table, then $f \preceq g$

i	$f(i)$	$g(i)$
1	2	3
2	3	3
3	1	2
4	2	4

(a) [3 marks] Find and show two functions f and g such that neither $f \preceq g$ nor $g \preceq f$ is true. (That is, find two incomparable functions.)

Solution (one of many):* *f(1) = 1, f(2) = 2, f(3) = 1, f(4) = 1
g(1) = 2, g(2) = 1, f(3) = 1, f(4) = 1

Marking: **Stating two incomparable functions: 3 marks**
Knowing what to do but not able to do it: 2 marks
Not clearly knowing what to do: 1 mark
No answer or completely wrong answer: 0 marks

(b) [9 marks] Prove that $P = (F, \preceq)$ is a poset.

Solution:

Reflexive: *For any function* f, $f(i) \leq f(i) \ \ \forall\, i \in S$

Therefore $f \preceq f$
Thus $\preceq$ *is reflexive*

Transitive: *Suppose* $f \preceq g$ *and* $g \preceq h$

$\Rightarrow \forall\, i \in S, f(i) \leq g(i)$ and $g(i) \leq h(i)$
$\Rightarrow \forall\, i \in S, f(i) \leq h(i)$
$\Rightarrow f \preceq h$
Thus $\preceq$ *is transitive*

Antisymmetric: *Suppose* $f \preceq g$ *and* $g \preceq f$

$\Rightarrow \forall\, i \in S, f(i) \leq g(i)$ *and* $g(i) \leq f(i)$
$\Rightarrow \forall\, i \in S, f(i) = g(i)$
$\Rightarrow f = g$
Thus $\preceq$ *is antisymmetric*

Marking: **3 marks for each property:**

for a sound argument	**3**
for an argument that demonstrates good understanding of the property	**2**
for an argument that demonstrates weak understanding of the property	**1**
for an argument that demonstrates no understanding of the property	**0**

(c) [3 marks] Either find the minimum element of P, or explain why P does not have a minimum element.

Solution:

Consider f(1) = 1, f(2) = 1, f(3) = 1, f(4) = 1

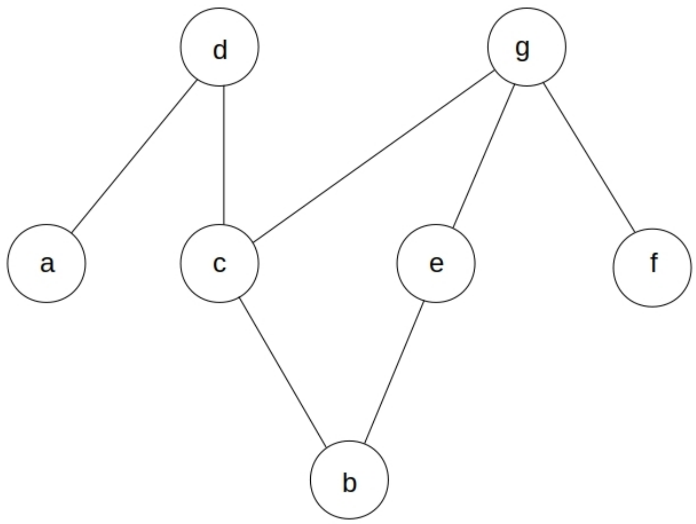
For any function g***,*** $f(i) \leq g(i) \;\; \forall \; i \in S$ ***, so*** $f \preceq g$

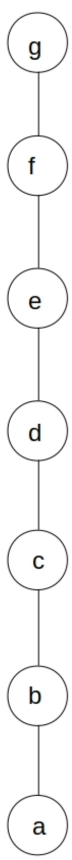
Thus f is the minimum element.

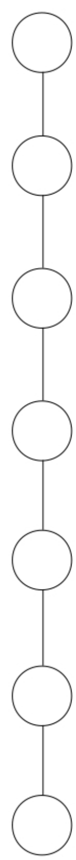
Marking:	**Stating the correct minimum :**	**3**
	Correctly defining "minimum" but not correctly finding it	**2**
	Stating there is no minimum	**1**
	No answer	**0**

Question 3: (10 Marks)

Here is a Hasse Diagram representing a poset. Also given is a linear extension of the poset. Find and show another linear extension such that the two linear extensions form a realizer for the poset.







Solution:

In the given linear extension, g is above a and d. It must be below them in the other extension, while still being above b, c, e, and f. This fixes g in the third position from the top.

Similarly, f must be below a,b,c,d,e and g – f must be in the bottom position.

a and d must be above g, so they must occupy the first and second positions. d must be above a, so d must be in the top position.

This leaves b, c and e, and the positions just above the bottom. b must be below the other two, so b must occupy the lowest of these three positions. In the given linear extension e is above c, so in the new extension c must be above e.

This gives the final order as

d
a
g
c
e
b
f

Marking:	**Stating the correct linear extension**	**10**
	Stating an incorrect linear extension	**1 ... 8**
	- depending how close they are to the correct answer	
	No answer or completely wrong answer	**0**

Students are not required to show how they found a correct solution.

Question 4 : (10 Marks)

Let $P = (X, \preceq)$ and $Q = (Y, \preceq^*)$ be two posets, where X and Y are finite sets, and $\preceq$ and $\preceq^*$ represent two different relations.

Suppose $X \cap Y \neq \emptyset$. Prove that $R = (X \cap Y, \preceq \cap \preceq^*)$ is a poset.

(Recall that a relation is a set of ordered pairs, and the intersection of two relations is the set of ordered pairs that belong to both relations.)

Solution:

***Reflexive:** $\preceq$ is reflexive, so it contains all reflexive pairs for elements of $X \cap Y$. The same is true of $\preceq^*$ Therefore $\preceq \cap \preceq^*$ contains all these pairs. Thus $\preceq \cap \preceq^*$ is reflexive.*

***Transitive:** Suppose a, b and c are all in $X \cap Y$, and (a, b) and (b, c) are both in $\preceq \cap \preceq^*$. Then both pairs are in $\preceq$, so $(a, c) \in \preceq$ (because $\preceq$ is transitive). Similarly, both pairs are in $\preceq^*$ so $(a, c) \in \preceq^*$. Thus $(a, c) \in \preceq \cap \preceq^*$. Therefore $\preceq \cap \preceq^*$ is transitive.*

***Antisymmetric:** Suppose (a,b) and (b,a) are both in $\preceq \cap \preceq^*$, with $a \neq b$ This implies both these pairs are in $\preceq$ ... which means $\preceq$ is not antisymmetric. CONTRADICTION. Therefore $\preceq \cap \preceq^*$ is antisymmetric*

Marking:

Reflexive:	**sound argument**	**3 marks**
Transitive:	**sound argument**	**4 marks**
Antisymmetric:	**sound argument**	**3 marks**

For each part, give part marks based on demonstrated understanding of the properties and the question.

Question 5 : (5 Marks)

In this question we will use the expression $a \preceq b$ where a and b are positive integers to mean "a divides b"

For example, $3 \preceq 3$, $13 \preceq 26$, $4 \preceq 12$ etc

Let S = {2, 3, 4, 8, 24}, and let P = (S, $\preceq$) You do not have to prove that P is a poset.

Let T = { {1} , {1,2} , {3} , {1,2,3} , {1,2,3,4,5} , {5} }, and let Q = (T, $\subseteq$) You do not have to prove that Q is a poset.

Show that P and Q are isomorphic. (Hint: posets are isomorphic if they have the same Hasse Diagram.)

Solution: This question contains an error

Marking: Any attempted answer 5 marks

S should be {2,3,4,5,12,60}

IF the question had been stated correctly, a correct answer would have been to draw the Hasse diagrams for the two posets and show that they are identical in structure (which is clear just from looking at them).