Proof that the Height of a RB Tree is in O(log n)

We have seen that the insert operation on a RB takes an amount of time proportional to the number of the levels of the tree (since the additional operations required to do any rebalancing require constant time at each level of the tree).

We have not yet shown that the number of levels in a RB tree is in O(log n). We need to do this so that we can truthfully claim that **search, insert** (and by extension, **delete**) are O(log n) operations on RB trees.

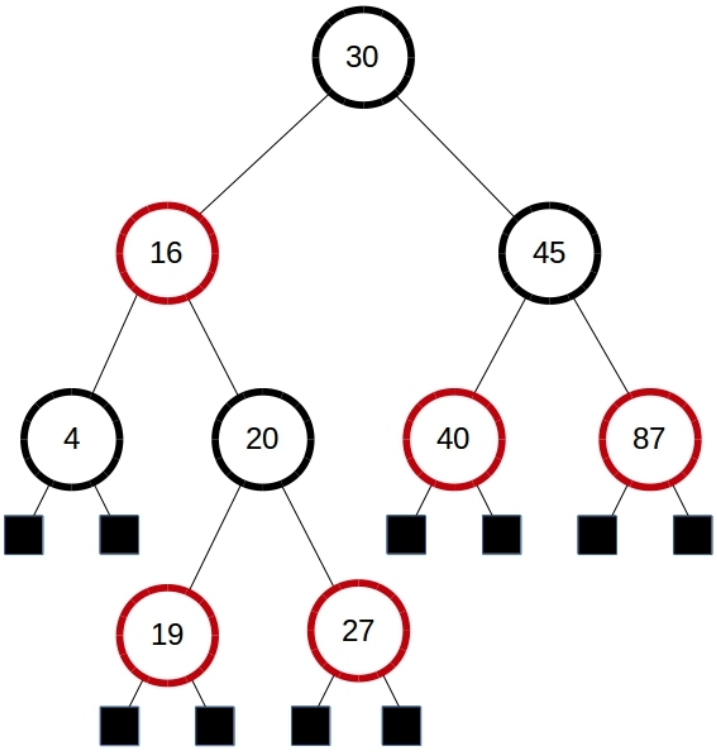
We will show that the height of a RB tree on n vertices is approximately 2*log n. In class I presented a simple structural proof of this claim:

Let T be a RB tree. Now compress T by absorbing each R vertex into its parent (ignoring the values). This creates a tree (call it T′) which is no longer a binary tree and which is pretty useless for practical purposes, but which is incredibly useful for proving the desired result about the height of T.

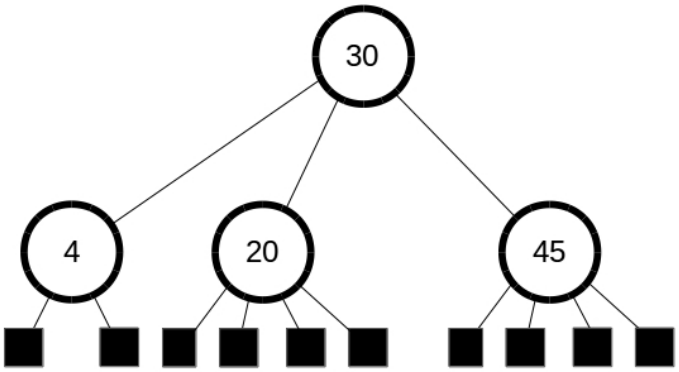
Here are some things we know about T′:

- all the leaves are at the same level (since the number of B vertices on every path down from the root of T to the leaves of T is the same, and we have just removed all the R vertices) and
- every internal vertex of T′ has 2, 3, or 4 children.

Here is a small example of T and T′:



The RB tree T. As an exercise, see if you can come up with a sequences of insertions that would create exactly this RB tree.



The modified tree T'. Each R vertex of T has been pulled up into its parent – this has the effect of raising some or all of the leaves to higher levels. We've lost some values but that's not important – we are not going to use this tree for anything except proving the desired property of the original RB tree. So we don't really care about the values at this point - all we care about is the structure of this tree.

Note that (as we claimed) all the leaves are now on the same level. Also note that the number of leaves has not changed (since they are all Black, none of them get absorbed into their parents).

Now, how many leaves does a RB tree have, if it contains n values? It may be a bit surprising to discover that a RB tree with n values *always* has exactly n+1 leaves. We can see this is true by an informal inductive argument. I will start with n = 1. A RB tree containing 1 value will consist of a single internal vertex with 2 leaves below it. There's our base case. Now consider what happens when we add a new value: one of the leaves is replaced by a new internal vertex with two leaves below it. So the number of values in the tree goes up by 1, and the number of leaves also goes up by 1. We will always have one more leaf than the number of values.

Let $m = n + 1$ be the number of leaves of T'.

Since the "branch-in" factor at each internal vertex of T' is at least 2 – that is, each internal vertex has at least two children, the number of vertices one level above the leaves is $\leq \frac{m}{2}$, and the level above that has $\leq \frac{m}{4}$ vertices, etc. decreasing by a factor of at least 2 on each

level until we reach the root. Clearly the number of levels before we reach the root is no more than $\log_2 m$. Thus in the original tree T, each path from the root to the leaves has $\leq \log_2 m$ B vertices (since the B vertices of T are all in T' as well). If we recreate T from T' by restoring the R vertices to their original positions, the lengths of paths from the root of T' to its leaves can at most double, due to Rule 4. Thus in T the longest path from the root to a leaf has length $\leq 2 * \log_2 m$.

Since m = n+1, we finally arrive at the conclusion that the longest path from the root of a RB tree to a leaf has length $\leq 2 * \log_2(n+1)$, where n is the number of values in the tree. Thus the height of the tree is in $O(\log n)$, and each of our search, insert and delete operations can be performed in $O(\log n)$ time.

Now, **finally**, we can conclude that for search, insert and delete operations, RB trees are better than sorted arrays.

The text gives a more mathematical proof of this result that I encourage you to read. I'll summarize it here. You are not required to know this proof, but it is an excellent example of a formal inductive argument about a data structure.

Let x be any vertex of a RB tree T. Define $bh(x)$ to be the number of Black vertices **below x** on any path from x to a leaf (by virtue of the properties of RB trees, $bh(x)$ is well defined and is independent of the specific path chosen). Define T_x to be the subtree rooted at x.

Define $IV(x)$ to be the number of internal vertices in T_x.

Claim: Let x be a vertex in a RB tree. Then $IV(x) \geq 2^{bh(x)} - 1$

Proof By Induction:

Base Case: Suppose $bh(x) = 0$ This means x is a leaf.

$2^{bh(x)} - 1 = 2^0 - 1 = 1 - 1 = 0$.

T_x has no internal vertices, so $IV(x) = 0 \geq 0 = 2^{bh(x)} - 1$. Thus the base case holds.

Inductive Hypothesis: Suppose for some $k \geq 0, bh(x) \leq k \Rightarrow IV(x) \geq 2^{bh(x)} - 1$ (i.e. assume the claim is true for all $bh(x)$ values $\leq k$)

Let x be a vertex with $bh(x) = k + 1$. This means x is an internal vertex, and thus x has

two children - call them y and z. There are two cases, based on the colour of x.

Case 1: x is Red. In this case, y and z must both be Black, and it is clear that $bh(y) = bh(z) = k$

Case 2: x is Black. In this case, each of y and z could be either Red or Black. If either of them is Black, it has $bh() = k$. If either of them is red, it must have two children with $bh() = k$.

In both Case 1 and Case 2, x has (at least) two descendants with $bh() = k$. By the Inductive Hypothesis stated above, the subtrees rooted at those descendants each have $\geq 2^k - 1$ internal vertices.

Thus $IV(x) \geq 2 * (2^k - 1) + 1$ ... the "+ 1" is there because x is an internal vertex of T_x.

Thus $IV(x) \geq 2^{(k+1)} - 1$

Thus $IV(x) \geq 2^{bh(x)} - 1$

(End of proof)

Now let T be a RB tree with n internal vertices and height h, and let r be the root of T. From the theorem we just proved, we get

$n \geq 2^{bh(r)} - 1$

so

$n + 1 \geq 2^{bh(r)}$

and

$\log(n + 1) \geq bh(r)$, which we simply turn around to write as $bh(r) \leq \log(n + 1)$

But clearly $h \leq 2 * bh(r)$, since the only difference between h and $bh(r)$ is the number of Red vertices on the paths from r to the leaves, and Red vertices can form no more than 1/2 of any path.

Combining these inequalities we get $h \leq 2 * log(n + 1)$

Thus the height of a RB tree with n internal vertices (ie a tree that stores n values) is

$$\leq 2 * \log(n+1)$$

All in all, I prefer the "push the red vertices up into their parent vertices" proof – I think it is far more intuitive.