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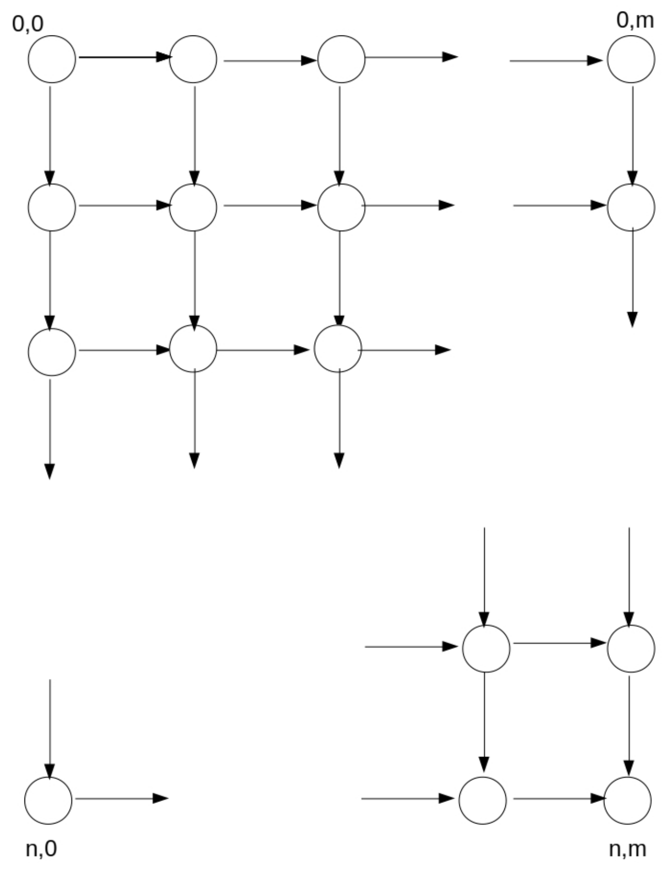
More Dynamic Programming

Consider this shortest path problem:

Given a rectangular grid with rows numbered 0,...,n and columns numbered 0,...,m

- with (0,0) in the top left corner and (n,m) in the bottom right corner
- with costs associated with all edges - notation: w(a,b : c,d) is the edge-cost (or weight) of the edge from (a,b) to (c,d)
- with all horizontal edges pointing to the right
- and with all vertical edges pointing downward



Our goal is to find the least-cost path from (0,0) to (n,m) (NB: this is one of the few instances where 0-based indexing is actually useful - it simplifies the calculation of the number of possible solutions.)

From the structure of the grid we can see that every path from (0,0) to (n,m) contains exactly n "down" edges and exactly m "right" edges. The total number of different paths from (0,0) to (n,m) is equal to the number of ways of interleaving the "down" and "right" edges. To get a sense of how many paths exist we can apply the following somewhat casual analysis:

For simplicity assume that n = m . Then the number of different paths is clearly $\binom{2n}{n}$ which is $\frac{(2n)!}{(n!n!)}$ This equals $\frac{((2n)(2n-1)...(n+1))}{(n(n-1)(n-2)...1)}$. Pairing each term in the numerator with the corresponding term in the denominator, we get

$$(\frac{2n}{n}) * (\frac{2n-1}{n-1}) * (\frac{2n-2}{n-2}) * ... * (\frac{n+2}{2}) * (\frac{n+1}{1})$$

Each term in this expression is clearly ≥ 2, so the product is $\geq 2^n$. Thus there are exponentially many potential paths - enumerating them all and choosing the least expensive (the BFI approach) is infeasible.

Instead, we make a clever observation: no matter what the optimum path looks like, the last step to (n,m) either comes downward from (n-1,m) or rightward from (n,m-1). So the Min_Cost path to (n,m) can be computed as follows

```
Min_Cost(n,m) = min { Min_Cost(n-1,m) + w(n-1,m : n,m),
                      Min_Cost(n,m-1) + w(n,m-1 : n,m)    }
```

And each of Min_Cost(n-1,m) and Min_Cost(n,m-1) is based in the same simple fashion on the vertices immediately above and immediately to the left. In fact, the same holds true for almost all of the vertices in the grid. The only exceptions are the vertices in row 0, which can only be reached from their immediate left, and the vertices in column 0, which can only be reached from immediately above.

So if we compute the Min_Cost values to all vertices in row 0 (since there is only one path to each of these vertices, this is easy) and to all vertices in column 0 (ditto), we can then compute the Min_Cost value for all other vertices, one row at a time, using the values we have previously calculated.

Thus the Min_Cost value for each vertex is computed exactly once, and requires a constant amount of work to compute. So the total time to compute all the Min_Cost values, including Min_Cost(n,m) is in O(n*m)

The key to formalizing the solution to this problem lies in correctly and precisely expressing it as a recurrence relation. We did part of this above, but we will now do it more completely.

Let Min_Cost(x,y) be the minimum total cost of a path from (0,0) to (x,y). It is important to observe (and it's also obvious) that Min_Cost(x,y) is well defined - there ***is*** a minimum cost path from (0,0) to (x,y)

```
Min_Cost(0,0) = 0
Min_Cost(i,0) = Min_Cost(i-1,0) + w(i-1,0 : i,0)          for i > 0
Min_Cost(0,j) = Min_Cost(0,j-1) + w(0,j-1 : 0,j)          for j > 0
Min_Cost(i,j) =   min { Min_Cost(i-1,j) + w(i-1,j : i,j),  for i,j > 0
                        Min_Cost(i,j-1) + w(i,j-1 : i,j)
                      }
```

We can store the Min_Cost values in a 2-dimensional array MC, with MC[i][j] = Min_Cost(i,j)

Now we can fill in the table one row at a time, working left to right in each row. Each element depends on either one or two previous elements, the values of which are already available. Thus each element is computed in constant time and the whole algorithm runs in O(n*m) time.

We could also fill the table one column at a time, with the same big-O complexity.

The final stage of the algorithm is the reconstruction of the details of the solution. MC[n][m] contains the cost of the minimum solution but we need to know what the optimal route is.

There are two approaches. One is to store extra information in the elements of MC. For example, when we compute MC[i][j] we can take note of whether we optimally reach vertex (i,j) on its incoming horizontal edge or its incoming vertical edge. We can record an "H" or

“V” in MC[i][j] along with the numeric cost of reaching the vertex (i,j). This increases the storage required for the MC array but it makes the reconstruction of the solution very easy: Starting at MC[n][m] we look at the direction tag. If it is “H” we know we reached (n,m) along its incoming horizontal edge so the previous vertex in the optimal solution is (n,m-1). On the other hand if the direction tag in MC[n][m] is “V” then the previous vertex in the optimal solution is (n-1,m)

The other way is to work our way back from (n,m) to (0,0) by examining the two candidates for the previous vertex:

```
if MC[n][m] == MC[n][m-1] + w(n,m-1 : n,m):
        previous vertex = (n,m-1)          # horizontal
else:
        previous vertex = (n-1,m)          # vertical
```

Now we repeat the same operation at the newly identified previous vertex, and so on until we get back to (0,0). This method requires no extra memory, but requires a bit more work than the other method.

Whichever method is used to reconstruct the optimal solution, we can see that each iteration takes constant time, and there are n+m iterations (since (n,m) is n+m steps away from (0,0)). Thus the reconstruction takes O(n+m) time.

Therefore the complexity of the entire algorithm is O(n*m + n + m) = O(n*m)

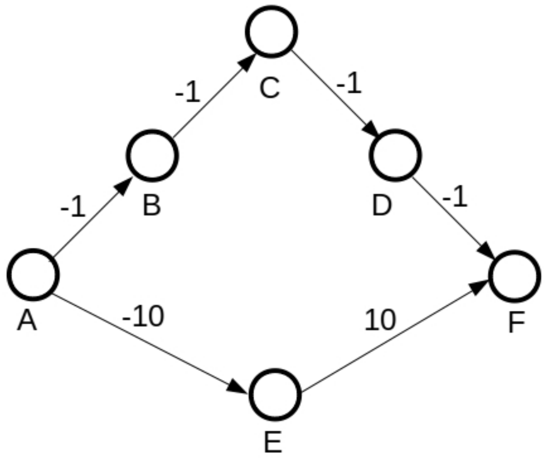
It is reasonable to ask why we need this algorithm at all? We already know that Dijkstra’s Algorithm will find the shortest path from any vertex to any other vertex. People sometimes forget that Dijkstra’s Algorithm has a limitation : all edge-weights must be ≥ 0. If any edge-weights are < 0, Dijkstra’s algorithm is not guaranteed to find the optimal solution.

However, our new algorithm ***is*** guaranteed to find the optimal solution for this problem, regardless of the edge-weights.

Actually it turns out for these very restricted and regularly structured grids, Dijkstra’s algorithm can be adapted to work even if some of the edges have negative weights. The trick is to increase all the edge weights by the absolute value of the most negative edge weight.

For example, if the most negative edge weight is -7, we increase ***all*** edge weights by 7. Now all the weights are non-negative, and it is relatively easy to show that the minimum weight path from (0,0) to (n,m) in the modified graph is also the minimum weight path in the original graph, and we can use Dijkstra's Algorithm to find it. (Exercise: compare the complexity of Dijkstra's Algorithm and the Dynamic Programming solution we just worked out.)

This trick worked for the grid-graph because all paths from (0,0) to (n,m) have the same number of edges. If that were not the case, it would not work. To demonstrate this, consider this graph. Like the grid-graph, it is directed and acyclic.



In this graph the minimum cost path from A to F is A-B-C-D-F with a cost of -4

But if we make all edge-weights non-negative by adding 10 to all of them, we find that the path A-B-C-D-F has total weight 36 while path A-E-F has total weight 20. Adding a constant amount to all the edge-weights has changed the optimal solution.